Inequalities, Oundle Session

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Much of section 2 of this sheet is from the sheet produced by Ben Green for the Trinity training session in 2000, and is worth working through when you have time.

1 Basic Inequalities.

In Olympiad Problems there is always a solution that does not require a great deal of knowledge, although it may take a good deal of ingenuity to find. In fact the jury try to avoid problems with particularly simple solutions that depend on obscure results or advanced techniques. For example, in most countries students do not learn until they go to university about how to use calculus to find maxima and minima of functions of several variables subject to constraints. Therefore IMO inequalities are only rarely easier to prove using those methods. You should make use of the fact that there is probably a clever but elementary solution using some of the following:

1. Simple but powerful facts like $x > y$ implies $x + z > y + z$; squares are non-negative; the third side of a triangle is not longer than the sum of the other two; the area of the union of several regions in the plane is at most the sum of the individual areas.

2. *The AM-GM Inequality.* If $a_1, \dots, a_n$ are non-negative reals then

$$\frac{a_1 + \cdots + a_n}{n} \geq (a_1 \dots a_n)^{\frac{1}{n}},$$

   with equality if and only if $a_1 = \dots = a_n$.

3. *The Cauchy-Schwarz Inequality.* Let $a_1, \dots, a_n$ and $b_1, \dots, b_n$ be two sets of real numbers. Then

$$(a_1 b_1 + \cdots + a_n b_n)^2 \leq \left(a_1^2 + \dots a_n^2\right)\left(b_1^2 + \cdots + b_n^2\right),$$

   with equality if and only if the a_i's and b_i's are *proportional*, i.e. $\mu a_i = \lambda b_i$ for some λ, μ. Note that this statement is true for *all* real numbers, not just non-negative ones. The Cauchy-Schwarz inequality essentialy says that the dot product of the two vectors $(a_1\, a_2\, \dots\, a_n)$ and $(b_1\, b_2\, \dots\, b_n)$ is at most the product of the lengths of the vectors. Solving olympiad problems using Cauchy-Schwarz is generally a matter of making a clever choice of the two vectors, for example so that one of the factors on the RHS is fixed.

4. *Jensen's Inequality.* Let f be a convex function on a certain interval $[a, b]$. This means that if you draw the graph of f then any chord joining two points lies above the graph. For example, $f(x) = x^2$ is convex on the whole interval $(-\infty, \infty)$. If you draw a picture then it is obvious that

$$f\left(\frac{c+d}{2}\right) \leq \frac{f(c) + f(d)}{2}$$

for any $c, d \in [a, b]$. This, essentially, is Jensen's inequality. In fact it is usually stated in the following more general form:

Let f be convex on $[a, b]$, let $x_1, \ldots, x_n \in [a, b]$, and let $p_1, \ldots, p_n \geq 0$ be weights with $p_1 + \ldots + p_n = 1$. Then

$$f(p_1 x_1 + \cdots + p_n x_n) \leq p_1 f(x_1) + \cdots + p_n f(x_n).$$

You can think of this as saying that the centre of mass of a collection of weights sitting on the graph of a convex function must lie above the graph. Or if you think of the weights as probabilities, then Jensen's inequality is saying that if X is a random variable taking values in an interval on which f is convex, then $\mathbb{E}(f(X)) \geq f(\mathbb{E}(X))$. Exercise: When do we get equality in Jensen's inequality?

The true power of this inequality only becomes apparent when you know calculus, because there is a rather nice condition for a function to be convex. Namely, a twice differentiable function f is convex on an interval $[a, b]$ precisely when the second derivative $f''(x)$ is non-negative throughout that interval.

Exercise *WAM-WGM* Show that the function $f(x) = e^x$ is convex on the whole of $(-\infty, \infty)$. Deduce the weighted AM-GM inequality: if $p_1, \ldots, p_n \geq 0$ with $p_1 + \ldots p_n = 1$ then for any $x_1, \ldots, x_n \geq 0$, we have

$$p_1 x_1 + \ldots p_n x_n \geq x_1^{p_1} x_2^{p_2} \ldots x_n^{p_n}$$

Exercise *The Power–Mean Inequality.* The α–power–mean of $y_1, \ldots, y_n > 0$ is

$$M_\alpha(y_1, \ldots, y_n) = \left(\frac{y_1^\alpha + \ldots + y_n^\alpha}{n}\right)^{1/\alpha}.$$

Apply Jensen's Inequality to the function $f(x) = x^{\alpha/\beta}$ to prove that if $\alpha > \beta > 0$ and $y_1, \ldots, y_n \geq 0$ then

$$M_\alpha(y_1, \ldots, y_n) \geq M_\beta(y_1, \ldots, y_n).$$

Now use the AM-GM inequality to extend this to the cases $\alpha > 0 > \beta$ and $0 > \alpha > \beta$. We get a hierarchy of "power-means" with the geometric mean corresponding to the exponent zero. Where do $\max(y_1, \ldots, y_n)$ and $\min(y_1, \ldots, y_n)$ fit in the hierarchy? The mean M_{-1} is called the *harmonic mean*:

$$\frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots \frac{1}{x_n}}.$$

The mean M_2 is called the quadratic mean. Use the Cauchy-Schwarz inequality to give another proof of the power-mean inequality in the special case $\alpha = 2\beta$.

Jensen's Inequality is often effective when applied in a geometric setting. For example the sine function is concave (i.e. $-sinx$ is a convex function of x) for $x \in [0, \pi]$. π means π radians, also known as 180 degrees. For an application of this, see problem 5 below.

5. Chebyshev's inequality. If $a_1 \leq a_2 \leq \ldots \leq a_n$ and $b_1 \leq b_2 \leq \ldots \leq b_n$ then

$$(a_1 + \ldots + a_n)(b_1 + \ldots + b_n) \leq n(a_1 b_1 + \ldots a_n b_n).$$

Exercise: prove Chebyshev's inequality, going via the following inequality:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (a_i - a_j)(b_i - b_j) \geq 0.$$

6. Hölder's inequality. This is a generalisation of the Cauchy-Schwarz inequality. If p and q are *conjugate* or *dual* real numbers, meaning that $p^{-1} + q^{-1} = 1$ and $p, q > 1$. Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be positive real numbers. Then

$$\left(a_1^p + \cdots + a_n^p\right)^{1/p} \left(b_1^q + \cdots + b_n^q\right)^{1/q} \ \geq \ a_1 b_1 + \cdots + a_n b_n.$$

Although no Olympiad problem is going to depend crucially on this inequality, knowing of its existence cannot hurt.

Exercise Using Hölder's Inequality deduce the Cauchy-Schwarz inequality. Deduce the case $p = 4$, $q = 4/3$ of Hölder's inequality from the Cauchy-Schwarz inequality.

2 Symmetry, homogeneity, constraints and substitutions

Some inequalities (like those featured above) are highly symmetric, and all variables appearing in them have equal importance. Others, like (`Exercise`) the inequality

$$x^8 + y^4 \ \geq \ 4x^2 y - 2,$$

are not. By the way in that inequality we need $y \geq 0$. As a general rule symmetry is a nice thing to have and should be kept, or created, if possible. However all rules are meant to be broken and there are some very symmetrical inequalities which can only be solved by singling out one variable and destroying the symmetry. Only experience will tell you whether to attempt this or not (although I'll try to say a bit more later). To illustrate what on earth I'm talking about, here is an example which can be done both ways.

`Example (David Monk).` Let a, b, c be non-negative reals, and suppose that $a + b + c \geq abc$. Prove that $a^2 + b^2 + c^2 \geq abc$.

`Solution 1.` We "note" the inequality

$$\left(a^2 + b^2 + c^2\right)^2 \ \geq \ abc(a + b + c),$$

which follows with room to spare by expanding out the left hand side –

$$\begin{aligned}\left(a^2+b^2+c^2\right)^2 &= a^4+b^4+c^4+a^2\left(b^2+c^2\right)+b^2\left(c^2+a^2\right)+a^2\left(a^2+b^2\right)\\ &\geq 0+0+0+a^2\cdot 2bc+b^2\cdot 2ca+c^2\cdot 2ab\\ &\geq abc(a+b+c).\end{aligned}$$

From this the result follows immediately.

`Solution 2.` Suppose the result is false. Since a, b, c have equal footing, we may assume without loss of generality that $a \geq b, c$. Now

$$a^2+2bc \;\leq\; a^2+b^2+c^2 \;<\; abc,$$

and so $a > 2$. Hence we may rearrange to get

$$\frac{a^2}{a-2} \;<\; bc.$$

`VERY IMPORTANT EXERCISE` *Why did we need $a > 2$ in order to make this deduction?*

Now remember we assumed $a \geq b, c$ and so

$$3a \;\geq\; a+b+c \;\geq\; abc,$$

and therefore $bc \leq 3$.

`Exercise` *Show that the two inequalities we have found for bc are not compatible. Hence, using Proof by Contradiction, complete the solution.*

`Insulting Exercise` *Which solution was symmetric, and which wasn't?*

`Exercise` *Show that in fact, under the assumptions of the problem, $(a^2+b^2+c^2)^2 \geq 3abc(a+b+c)$.*

2.1 Homogeneity

An inequality involving (multidimensional) polynomials is said to be *homogeneous* if all the polynomials have the same degree. For example the AM-GM and Cauchy-Schwarz Inequalities are both homogeneous. So to kill an inequality off using one of those, we must first get it into a homogeneous form, for example by making use of non-homogeneous constraints. Here are two examples.

`Example (BMO Round 2 1999, Q3).` Let p, q, r be non-negative reals with sum 1. Show that

$$7(pq+qr+rp) \;\leq\; 2+9pqr.$$

`Solution.` The inequality is symmetric, which we like, but not homogeneous. Indeed there are terms of all sorts of degrees, like 0,2 and 3. However if we play around for a while using the condition $p+q+r=1$ (which allows us to replace a degree 0 term by a degree 1 term), we find that

$$2+9pqr-7(pq+qr+rp) \;=\; 3\left(p^3+q^3+r^3\right)-(p+q+r)\left(p^2+q^2+r^2\right).$$

Satisfy yourself of this. Now the right hand side is homogeneous and quite friendly-looking: all the 7's and 9's have disappeared.

`Exercise` *Use the Cauchy-Schwarz Inequality to prove that*

$$3\left(p^3+q^3+r^3\right) \;\geq\; (p+q+r)\left(p^2+q^2+r^2\right).$$

A case in which one can easily make a non-homogeneous inequality homogeneous is in the presence of a multiplicative constraint. For example, given the condition $abc = 1$, it can often be helpful to set $a = u/v$, $b = v/w$ and $c = w/u$. Then there are no constraints on u, v, w and the new inequality will be homogenous.

`Example (FST 1994 Q1)` Let $a, b, c \geq 0$. Prove that

$$a^5+b^5+c^5 \;\geq\; 5abc(b^2-ac).$$

`Hints for Solution` This is a really superb (and quite hard) problem which I shall not spoil by solving completely. Observe that the inequality is homogeneous of degree 5, but is not symmetric as the variable b has a distinguished rôle.

`Exercise` *Show that the inequality we want follows from the "dehomogenised" inequality*

$$x^5+1+y^5 \;\geq\; 5xy(1-xy)$$

obtained by putting $b = 1$. *(Hint: find a suitable substitution).*

It is a general principle that we may assign a non-zero value of our choice to *one* of the variables in a homogenous inequality. This is seldom useful, but here it turns out to be pretty much essential.

A little while ago I promised some general remarks on when to break symmetry and/or homogeneity. So.....

When to try breaking symmetry or homogeneity

– If there are not too many variables (say just 2 or 3)

– If one of the variables is obviously different from all the others

– If equality can occur for some non-symmetric values of the variables (although you might not know whether this is the case while you're trying to solve the problem)

– If nothing else is working!

2.2 Check There's Hope.

The solution of FST 1994 Q1 reminds me of a very important point. Quite often in an inequalities problem one might proceed as follows. We want to prove $A \geq B$, and we know how to prove $A \geq C$ and $D \geq B$. If only we could get $C \geq D$....... It may seem obvious, but if you use this kind of approach it is worth investing some time checking whether $C \geq D$ has any hope of being true. One way of doing this might be to substitute in a few values, particularly in the neighbourhood of where we expect equality to occur in $A \geq B$.

A situation in which there *cannot* be any hope is where the conditions for equality in $A \geq C$ are not those for equality in $A \geq B$.

`Exercise` *Why is the following attempt to solve FST 1994 Q1 doomed to failure? "Using AM-GM, we have* $a^5 + b^5 + c^5 \geq 3(abc)^{\frac{5}{3}}$*...."*

2.3 Without Loss Of Generality.....

This is an oft-used phrase in the world of inequalities, and in fact it has already featured in these notes. However it must be used with care.

`Exercise` *Suppose we're trying to prove that* $a^2b + b^2c + c^2a \geq 3$ *when* $abc = 1$ *and* a, b, c *are positive. Why would it be a very bad idea to start the solution with the phrase "Without loss of generality,* $a \geq b \geq c$*"?*

2.4 Calculus

This has always been a somewhat dirty word in Olympiad circles. There are a number of reasons for not going straight into an Olympiad problem using calculus.

– The problem has almost certainly been designed so that there is no remotely nice calculus solution.

– Calculus only finds *local* maxima and minima. If the problem involves maximising some quantity over, say, the cube $0 \leq a, b, c \leq 1$ then naive calculus will not tell us anything about maxima on the boundary of the region.

– The conditions for a stationary value to be a maximum/minimum become very nasty in problems with several variables, as one finds objects like saddle-points making an appearance. If the inequality is one with a constraint (like $a^2+b^2+c^2 = 1$) and you have used Lagrange Multipliers, the conditions are even worse.

That having been said, if you are really stuck then calculus might well at least give you an idea of *what* the maximum is and *where* it is attained. In FST 1994 Q1 I was very stuck indeed, not even having a reasonable guess about where equality occurred, and 2-variable calculus rescued me (although the final solution was purely algebraic). If you know where equality occurs then you can often craft your solution around this.

`Exercise` *Where does equality occur in that problem?*

It might also happen that you can reduce a given inequality to another inequality in only *one* variable, and in this situation calculus is often a good tool to use.

`Hard Exercise` *Use calculus sparingly to deduce the following extension of IMO 1984 Q1. Let* $a, b, c \geq 0$ *satisfy* $a + b + c = 1$*, and let* $\lambda > 0$*. Show that*

$$ab + bc + ca - \lambda abc \;\leq\; \max\left\{\frac{1}{4}, \frac{1}{27}(9 - \lambda)\right\}.$$

2.5 Variational Methods

Here is another topic of a slightly contentious nature. What I am talking about is the following kind of argument (although the following example is a little contrived): `Example` Minimise $a^2 + b^2 + c^2$ subject to $a, b, c \geq 0$ and $a + b + c = 1$.

`Solution.` Suppose we replace (a, b, c) by $\left(\frac{a+b}{2}, \frac{a+b}{2}, c\right)$ (observe that the sum of these numbers is still 1). Then $a^2 + b^2 + c^2$ decreases by

$$a^2 + b^2 - 2\left(\frac{a+b}{2}\right)^2 \;=\; \frac{1}{2}(a-b)^2 \;\geq\; 0.$$

The same is true if we replace (a, b, c) by $\left(\frac{a+c}{2}, b, \frac{a+c}{2}\right)$ or $\left(a, \frac{b+c}{2}, \frac{b+c}{2}\right)$. If we iterate such replacements then (a, b, c) can be made to approach $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ (*Why?*). Therefore the minimum value is $\frac{1}{3}$, and this occurs when $a = b = c = \frac{1}{3}$.

`Exercise` *What is wrong with the following argument? "Let (a, b, c) be the point at which the minimum occurs. If $a \neq b$ then replace (a, b, c) by $\left(\frac{a+b}{2}, \frac{a+b}{2}, c\right)$. Then $a^2 + b^2 + c^2$ decreases, contradiction. Therefore the minimum is $\frac{1}{3}$, and this occurs when $a = b = c = \frac{1}{3}$".*

Whether or not you like this style of argument is a matter of taste. However it is unlikely that an IMO question would be set which could be done in this way (although there was a solution of BMO Round 2 1999 Q3 along these lines). Note that it was essential that our "perturbation" of (a, b, c) to $\left(\frac{a+b}{2}, \frac{a+b}{2}, c\right)$ left the sum of the three elements invariant. This method is therefore best suited to problems with a linear constraint (like $a + b + c = 1$).

2.6 Induction

If the problem at hand has lots of variables, why not try induction? There need not be an "n" anywhere in the statement of the problem for this kind of idea to work.

`Exercise` *Let $a, b, c, d, e, f \geq 0$.* Prove that

$$\frac{ab}{a+b} + \frac{cd}{c+d} + \frac{ef}{e+f} \;\leq\; \frac{(a+c+e)(b+d+f)}{a+b+c+d+e+f}.$$

`Hint:` *Why have I put this problem here?*

2.7 Geometric Inequalities

Everything we've said above still applies, although there are a few more "basic" inequalities and techniques to add to your armoury. I won't say anything about it here except to remind you that if a, b, c are the sides of a triangle, then the triangle inequality applies in three ways. Those contraints are hard to use, but the substitution $a = p + q$, $b = q + r$, $c = r + p$ reduces these contstraints to $p, q, r \geq 0$. Finally, it's worth bearing in mind that geometrical inequalities are often invented by taking an equality and observing that one or more squares must be non-negative.

3 Bonus Problems

Not all of these problems are meant to be easy. Nor are they meant to provide an illustration of everything I said above.

1. Let n be a positive integer. Let $d(n)$ denote the number of divisors of n, and let $\sigma(n)$ be the sum of these divisors (including n itself). Prove that
$$\frac{\sigma(n)}{d(n)} \geq \sqrt{n}.$$
2. Let $a, b, c \geq 0$ have product 1. Prove that
$$a^5 + b^5 + c^5 \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$
3. **(David Monk)** Let a, b, c, d be real numbers with $ad - bc = 1$. Prove that
$$a^2 + b^2 + c^2 + d^2 + ac + bd \geq \sqrt{3}.$$
4. Let $0 \leq x, y, z \geq 1$. Show that
$$x^2 + y^2 + z^2 \leq x^2 y + y^2 z + z^2 x + 1.$$
5. Let ABC be a triangle and P an interior point. Show that at least one of the angles PAB, PBC, PCA is less than or equal to 30 degrees.
6. Let a, b, c be the sides of a triangle. Suppose furthermore that this triangle has perimeter 1. Prove that
$$\frac{23}{216} < a^2 b + b^2 c + c^2 a < \frac{1}{8}.$$
7. Let $a, b, c, d, e > 0$ be real numbers with product 1. Show that
$$\frac{a + abc}{1 + ab + abcd} + \frac{b + bcd}{1 + bc + bcde} + \frac{c + cde}{1 + cd + cdea} + \frac{d + dea}{1 + de + deab} + \frac{e + eab}{1 + ea + eabc} \geq \frac{10}{3}.$$
8. Let x, y, z be positive reals. Show that
$$x^4 + y^4 + z^4 + \frac{(x + y + z)^4}{27} \geq 2(x^2 y^2 + y^2 z^2 + z^2 x^2).$$
9. Let $p, q, r > 0$. Show that
$$(p + q + r)\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right) \geq 9 + 3\left(1 - \frac{p}{q}\right)^{2/3}\left(1 - \frac{q}{r}\right)^{2/3}\left(1 - \frac{r}{p}\right)^{2/3}.$$

10. Determine the least value of

$$\frac{a}{b+2c} + \frac{b}{c+2a} + \frac{c}{a+2b}$$

over all positive real numbers a, b, c.

11. Show that all the stationary points of a (complex) polynomial f lie in the convex hull (in the compex plane) of the set of roots of f.

12. Let P be a point in the interior of triangle ABC. Let d, e, f be the distances from P to the sides a, b, c of the triangle, respectively. Show that the minimum value of

$$\frac{a}{d} + \frac{b}{e} + \frac{c}{f}$$

occurs when P is at the incentre.

13. Let S be a finite set of points in $\mathbb{R}^3$. Let S_x, S_y, S_z be the projections of S onto the yz, xz, xy planes respectively. Show that

$$|S|^2 \leq |S_x|.|S_y|.|S_z|,$$

where $|A|$ stands for the number of elements in the set A.